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Equations for the continuous estimation of the perturbations of dynamical systems☆

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Abstract

A dynamical system acted upon by external perturbations is considered. It is assumed that the phase state of the system (or a part of it) is observed with certain errors. The problem is to construct differential equations for estimating (reconstructing) the perturbations using measurement data. Unlike in papers in which cases of discrete instants of the observations are analysed, the continuous case is considered for which differential equations of an auxiliary system are derived, the controls in which are approximations of an unknown input. The general constructions are illustrated by means of an example.

1. Introduction: Formulation of the problem

A dynamical system is specified which is described by the non-linear differential equation

$$\dot{x}(t) = f(t, x(t)) + Bu(t), \quad t \in [0, T], \quad x(0) = x_0, \tag{1.1}$$

where t is the time, $x \in R^n$ is the phase vector of the system, $u(t) \in R^m$ is a perturbation, B is an $(n \times m)$-dimensional matrix and f is a $(n \times n)$-dimensional matrix function which is continuous with respect to t and Lipschitzian with respect to x. The trajectory of the system $x(\cdot)$ depends on an input action (perturbation) $u(\cdot)$ which varies with time. This action, as well as the trajectory, are not specified in advance. It is assumed that observations (measurements) of the phase state of system (1.1) are carried out continuously, as a result of which the vectors $\xi^h(t) \in R^{n_1}$, $n_1 \leq n$ with the properties

$$\left|\xi^h(t) - \{x(t)\}_{n_1}\right|_{n_1} \leq h, \quad t \in [0, T]. \tag{1.2}$$

are determined. The quantity $h \in (0, 1)$ characterizes the accuracy of the measurement, $\{x\}n_1$ is a vector composed of the first n_1 coordinates of the vector x and $|\cdot|_n$ is a Euclidean norm in the space R^n.

The problem of continuous estimation involves constructing an algorithm for the approximate recovery of the unknown perturbation $u(\cdot)$, which possesses the properties of dynamicity and stability. The property of dynamicity means that the current values of the approximation to the unknown perturbation are processed in real time while the property of stability means that the approximation is as accurate as may be desired when the error in the observation channel is fairly small. In this case, at an instant t it is permissible to use the results of an observation $\xi^h(t)$ in an interval $[0, t]$.

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The problem being discussed belongs to the class of inverse problems in the dynamics of control systems. Similar problems have been investigated earlier (see Refs. 1–3, for example) An approach to the solution of the problem of the dynamic recovery of the input to a finite-dimensional system of the form of (1.1) was proposed in Ref. 4 in the case when a convex, bounded and closed set $P \subset R^m$ (the set of "instantaneous" constraints) is specified with the property $u(t) \in P$ when $t \in [0, T]$ almost everywhere. This approach, which has been developed further in a number of papers (see the Refs. 5–7 and the review Ref. 8), is based on a combination of the principle of positional control, which is known in the theory of guaranteed control, with the model in Ref. 9 and, also, with one of the basic methods in the theory of ill-posed problems,[10] that is, with the smoothing functional method (Tikhonov's method). Note that problems of the dynamic reconstruction of inputs within the framework of the approach developed by Osipov and Kryazhimskii[5] have been studied for the case of observation of the phase state at discrete instants of time.[5–7,11] An algorithm for solving the problem of estimation (reconstruction) in the case of continuous measurement is indicated below based on the well-known ideas in Refs. 1,4,11.

2. Estimation equations: The case of measurement of all of the coordinates

We will first consider the case of measurement of all of the phase coordinates of system (1.1). Actually, we shall assume that $n_1 = n$. Consequently, the results of the observations are the n-dimensional vectors $\xi^h(t)$ with the properties

$$|\xi^h(t) - x(t)|_n \le h, \quad t \in [0, T].$$

Suppose L is the Lipschitz constant of the function f, that is,

$$|f(t, x_1) - f(t, x_2)|_n \le L|x_1 - x_2|_n, \quad \forall t \in [0, T], \quad x_1, x_2 \in R^n.$$

We introduce the auxiliary function $\alpha(h) \in (0, 1)$, which possesses the following property

$$\alpha(h) \to 0, \quad h^{2/3}\alpha^{-1}(h) \to 0 \text{ when } h \to 0. \tag{2.1}$$

The function plays the role of a regularizer. It is clear from Theorem 1, presented below, that a control system of the form

$$\dot{w}^h(t) = f(t, w^h(t)) + B\upsilon^h(t) + \nu^h(t), \quad t \in [0, T] \tag{2.2}$$

with an initial state $w^h(0) = \xi^h(0)$ can be taken as the continuous estimation equation.

We introduce the notation

$$\Delta^h(t) = \xi^h(t) - w^h(t), \quad \Delta_*(t) = u_*(t) - \upsilon^h(t).$$

We specify the controls $\upsilon^h(t)$ and $\nu^h(t)$ in system (2.2) as follows:

$$\upsilon^h(t) = \alpha^{-1}B'\Delta^h(t), \quad \nu^h(t) = L\Delta^h(t). \tag{2.3}$$

A prime denotes transposition.

Suppose $u_*(\cdot) = u_*(\cdot; x(\cdot))$ is an element of the set $U(x(\cdot))$ of minimal $L_2([0, T]; R^m)$-norm and $U(x(\cdot))$ is the set of all controls $u(\cdot)$, compatible with the input $x(\cdot)$, that is,

$$U(x(\cdot)) = \{u(\cdot) \in L_2([0, T]; R^m)\}:$$

$$x(t) = x_0 + \int_0^t \{f(\tau, x(\tau)) + Bu(\tau)\}d\tau, \quad \forall t \in [0, T].$$

We note that the set $U(x(\cdot))$ is convex and closed in $L_2([0, T]; R^m)$. The element $u_*(\cdot)$ is therefore defined and unique.

Theorem 1. *Suppose conditions (2.1) are satisfied. Then, the convergence*

$$\upsilon^h(\cdot) \to u_*(\cdot) \text{ in } L_2([0, T]; R^m) \text{ when } h \to 0.$$

holds.

By virtue of a well-known theorem ((Ref. 7, Theorem 2.1), to prove Theorem 1 it is sufficient to establish the correctness of the lemma which is formulated next.

We first introduce the notation

$$I_{km}(t) = \int_0^t \left|\upsilon^h(\tau)\right|_m^k d\tau, \quad J_{km}(t) = \int_0^t \left|u_*(\tau)\right|_m^k d\tau$$

$$K_{kn}(t) = \int_0^t \left|\mu_h(\tau)\right|_n^k d\tau, \quad \mu_h(t) = x(t) - w^h(t).$$

Lemma 1. *Constants d_0 and d_1 can be found (in an explicit form) which are such that the inequalities*

$$\left|\mu_h(t)\right|_n^2 \le d_0(h^{2/3} + \alpha), \quad t \in [0, T]; \quad I_{2m}(T) \le J_{2m}(T) + d_1 h^{2/3}\alpha^{-1}; \quad \alpha = \alpha(h).$$

hold.

Proof. By virtue of relations (1.2) and (2.3), the inequality

$$\left|\upsilon^h(t)\right|_m^2 \le 2b^2\alpha^{-2}(h^2 + \left|\mu_h(t)\right|_n^2), \quad t \in [0, T],$$

holds, where $b = |B'|$ is the Euclidean norm of the matrix B'. In this case,

$$I_{2m}(t) \le 2b^2\alpha^{-2}K_{2n}(t) + c_1 h^2\alpha^{-2}. \tag{2.4}$$

It is also clear that the inequality

$$(B\Delta_*(t), \mu_h(t))_n \le (B\Delta_*(t), \Delta^h(t))_n + bh\varrho_h(t) \text{ при п.в. } t \in [0, T]$$

$$\varrho_h(t) = \left|u_*(t)\right|_m + \left|\upsilon^h(t)\right|_m.$$

holds. The symbol $(\cdot;\cdot)_n$ denotes a scalar product in R^n.

Next, multiplying the left- and right-hand sides of the equality

$$\dot{x}(t) - \dot{w}^h(t) = f(t, x(t)) - f(t, w^h(t)) + B\Delta_*(t) - \mathrm{v}^h(t)$$

by $\mu_h(t)$, we shall have

$$\frac{1}{2}\frac{d\left|\mu_h(t)\right|_n^2}{dt} \le (B\Delta_*(t), \mu_h(t))_n + L\left|\mu_h(t)\right|_n^2 - L(\Delta^h(t), \mu_h(t))_n \le$$
$$\le (B\Delta_*(t), \Delta^h(t))_n + bh\varrho_h(t) + L\left|\mu_h(t)\right|_n^2 - L(\Delta^h(t), \mu_h(t))_n.$$

Consequently,

$$\frac{d\left|\mu_h(t)\right|_n^2}{dt} + \alpha\left\{\left|\upsilon^h(t)\right|_m^2 - \left|u_*(t)\right|_m^2\right\} \le -2(\upsilon^h(t), B'\Delta^h(t))_m + \alpha\left|\upsilon^h(t)\right|_m^2 +$$
$$+ 2(u_*(t), B'\Delta^h(t))_m - \alpha\left|u_*(t)\right|_m^2 + 2bh\varrho_h(t) + 2Lh\left|\mu_h(t)\right|_n. \tag{2.5}$$

Note that the control $\upsilon^h(t)$ the form of (2.3) is such that

$$\upsilon^h(t) = \arg\min\{\alpha|\upsilon|_m^2 - 2(B'\Delta^h(t), \upsilon)_m : \upsilon \in R^m\}. \tag{2.6}$$

From relation (2.5) we obtain, by virtue of Eq. (2.6),

$$\varepsilon_h(t) \le \varepsilon_h(0) + 2bh\int_0^t \varrho_h(\tau)d\tau + 2LhK_{1n}(t), \tag{2.7}$$

where

$$\varepsilon_h(t) = |\mu_h(t)|_n^2 + \alpha(I_{2m}(t) - J_{2m}(t)).$$

In view of the inclusion $u_*^{\backslash}(\cdot) \in L_2([0, T]; R^m)$, we have

$$\int_0^T 2bh|u_*(\tau)|_m d\tau \le c_2 h.$$

Moreover,

$$2LhK_{1n}(t) \le L^2 h^{2-\gamma} + h^\gamma t K_{2n}(t), \quad \gamma \in (0, 1).$$

Hence, in this case and from inequality (2.7) we obtain

$$\varepsilon_h(t) \le \varepsilon_h(0) + c_3(h + h^\beta + h^{2-\gamma} + h^{2-\beta} I_{2m}(t)) + h^\gamma t K_{2n}(t), \quad \beta \in (0, 1). \tag{2.8}$$

In turn, from relation (2.8) by virtue of inequality (2.4), we derive (since $\varepsilon_h(0) = h^2$, $h^{2-\gamma} \le h^\beta$)

$$\varepsilon_h(t) \le c_4 F_1(\alpha) + c_5 F_2(\alpha) K_{2n}(t), \tag{2.9}$$

where

$$F_1(\alpha) = h^\beta + h^{4-\beta}\alpha^{-2}, \quad F_2(\alpha) = h^\gamma + h^{2-\beta}\alpha^{-2}.$$

The estimate

$$|\mu_h(t)|_n^2 \le c_6(F_1(\alpha) + \alpha) + c_5 F_2(\alpha) K_{2n}(t) \le c_6(F_1(\alpha) + \alpha)\exp\{c_5 t F_2(\alpha)\}, \quad t \in [0, T] \tag{2.10}$$

therefore follows from inequality (2.9) (Gronwall's lemma has also been used).

Suppose $\beta \in (0, 1)$ is a constant such that

$$h^{2-\beta}\alpha^{-2} \le \text{const}, \quad h \in (0, 1). \tag{2.11}$$

Then,

$$|\mu_h(t)|_n^2 \le c_7(h^\beta + \alpha). \tag{2.12}$$

From relations (2.9), (2.11) and (2.12), we obtain

$$\varepsilon_h(t) \le c_4 F_1(\alpha) + c_8 F_2(\alpha)(h^\beta + \alpha) \le c_9(F_3(\alpha) + h^{\beta+\gamma} + h^\gamma \alpha),$$

where

$$F_3(\alpha) = h^\beta + h^{2-\beta}\alpha^{-1} + h^2\alpha^{-2}.$$

Putting $\gamma = \beta$, from the last inequality we obtain

$$\varepsilon_h(t) \le c_{11} F_3(\alpha). \tag{2.13}$$

In this case

$$\alpha I_{2m}(t) \le \alpha J_{2m}(t) + c_{11} F_3(\alpha). \tag{2.14}$$

Assuming that $\beta = 2/3$, $h^{2/3}\alpha^{-1} \in (0, 1)$, from inequality (2.14) we obtain

$$I_{2m}(T) \le J_{2m}(T) + c_{12}\alpha^{-1} F_3(\alpha) \le J_{2m}(T) + c_{13} h^{2/3}\alpha^{-1}. \tag{2.15}$$

The correctness of the lemma follows from inequalities (2.12) and (2.15).

Under certain additional conditions, an estimate of the rate of convergence (see Lemma 3 below) can be written out. The following lemma is required to prove this estimate.

Lemma 2 [5]. *Suppose* $u(\cdot) \in L_\infty([0; T]; R^n)$, $\upsilon(\cdot)$ *is a function of bounded variation and*

$$\left|\int_0^t u(\tau)d\tau\right|_n \le \varepsilon, \quad |\upsilon(t)|_n \le K, \quad \forall t \in [0, T].$$

Then,

$$\left|\int_0^t (u(\tau), \upsilon(\tau))_n d\tau\right| \le \varepsilon(K + \mathrm{var}([0, T]; \upsilon(\cdot))), \quad \forall t \in [0, T]$$

($\mathrm{var}([0, T]; \upsilon(\cdot))$ *is the variation of the function* $\upsilon(\cdot)$ *in the interval* $[0,T]$).

Lemma 3. *Suppose* $m = n$, B *is an invertible* $(n \times n)$*-matrix and* $u_*(\cdot)$ *is a function of bounded variation. The following estimate of the rate of convergence of the algorithm then holds*

$$|u_*(\cdot) - \upsilon^h(\cdot)|^2_{L_2([0, T]; R^n)} \le K\{h^{1/3} + h^{2/3}\alpha^{-1} + \alpha^{1/2}\}; \quad \alpha = \alpha(h).$$

Proof. Note that the inequality

$$\left|\int_{t_1}^{t_2} B\Delta_*(t)dt\right|_n = \left|\int_{t_1}^{t_2}[\dot{x}(\tau) - \dot{w}^h(\tau) - f(\tau, x(\tau)) + f(\tau, w^h(\tau)) - \nu^h(\tau)]d\tau\right|_n \le$$
$$\le |\mu_h(t_2) - \mu_h(t_1)|_n + K_1\int_{t_1}^{t_2}(h + |\mu_h(\tau)|_n)d\tau,$$

holds, for any $t_1, t_2 \in [0, T]$, $t_1 < t_2$ where, as above, $\mu_h(t) = x(t) - w^h(t)$. Moreover, by virtue of Lemma 1,

$$|\mu_h(t)|_n \le K_2(h^{2/3} + \alpha)^{1/2}.$$

From this, we deduce that

$$\left|\int_{t_1}^{t_2} B\Delta_*(t)dt\right|_n \le K_3(h^{2/3} + \alpha)^{1/2}.$$

Using Lemma 2 and relation (2.15), we obtain

$$|\Delta_*(\cdot)|^2_{L_2([0, T]; R^n)} \le 2|u_*(\cdot)|^2_{L_2([0, T]; R^n)} - 2\int_0^T (u_*(\tau), \upsilon^h(\tau))_n d\tau + c_{13}h^{2/3}\alpha^{-1} =$$
$$= 2\int_0^T ((B')^{-1}u_*(\tau), B\Delta_*(\tau))_n d\tau + c_{13}h^{2/3}\alpha^{-1} \le K\{(h^{2/3} + \alpha)^{1/2} + h^{2/3}\alpha^{-1}\}.$$

Remarks.

1°. If it is assumed that $\alpha = \alpha(h) = h^{4/9}$, then, when the conditions of Lemma 3 are satisfied, we have

$$\sup_{t \in [0, T]}|\mu_h(t)|_n \le K_0 h^{2/9}, \quad |\Delta_*(\cdot)|_{L_2([0; T]; R^n)} \le K_1 h^{1/9}.$$

2°. It can be established in a similar way to the well-known approach in Ref. 12 that the assertion of Lemma 3 is also true if $m < n$ and the rank of the matrix B is equal to m.

3. Estimation equations: The case of measurement of part of the coordinates

We will now consider the case of the measurement of part of the coordinates of the phase vector ($n_1 < n$). Suppose y is a vector consisting of the first n_1 coordinates of the vector x, and z is a vector consisting of the remaining $n - n_1$ coordinates of the vector x. Hence, $x = (y, z)$. Suppose

$$f(t,x) = f(t,y,z) = \begin{pmatrix} f_1(t,y) + Cz \\ f_2(t,y,z) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ D \end{pmatrix},$$

where $n_1 > n/2$, the rank of the $(n_1 \times (n - n_1))$-dimensional matrix C is equal to $n - n_1$ and the matrix D has the dimensions $(n - n_1) \times m$. In this case, system (1.1) can be rewritten in the form

$$\dot{y}(t) = f_1(t, y(t)) + Cz(t), \quad \dot{z}(t) = f_2(t, y(t), z(t)) + Du(t). \tag{3.1}$$

At the same time, inequalities (1.2) take the form

$$|\xi^h(t) - y(t)|_{n_1} \le h, \quad t \in T, \quad \xi^h(t) \in R^{n_1}. \tag{3.2}$$

Note that the linearity with respect to z in the first equation of system (3.1) is of fundamental importance, since the technique described in the preceding section enables one to recover a linear input. At the same time, it is possible to treat the case when the matrix C depends on y in a similar manner to that indicated earlier in Refs. 4,5.

We shall also assume that, at the initial instant of time, the whole of the initial state of the system is measured, that is, the vector $\xi_0^h = (\xi^h(0), \xi_1^h(0)) \in R^n$ is determined such that $|\xi_0^h - x(0)|_n \le h$. As the equation for continuous estimation, we take the system

$$\begin{aligned} \dot{w}_1^h(t) &= f_1(t, w_1^h(t)) + C\upsilon_1^h(t) + \nu_1^h(t) \\ \dot{w}_2^h(t) &= f_2(t, w_1^h(t), w_2^h(t)) + D\upsilon_2^h(t) + \nu_2^h(t) \end{aligned} \tag{3.3}$$

with controls of the form

$$\begin{aligned} \upsilon_1^h(t) &= h^{-4/9} C'\Delta_1^h(t), \quad \nu_1^h(t) = L\Delta_1^h(t) \\ \upsilon_2^h(t) &= \alpha^{-1} D'\Delta_2^h(t), \quad \nu_2^h(t) = 2L\Delta_2^h(t), \end{aligned} \tag{3.4}$$

where

$$\Delta_1^h(t) = \xi^h(t) - w_1^h(t), \quad \Delta_2^h(t) = \upsilon_1^h(t) - w_2^h(t)$$

and $\alpha = \alpha(h)$ is an auxiliary parameter. For the initial state of system (3.3), we take ξ_0^h, that is,

$$w_1^h(0) = \xi^h(0), \quad w_2^h(0) = \xi_1^h(0).$$

Theorem 2. *Suppose*

$$\alpha = \alpha(h) = h^{1/18} \tag{3.5}$$

in expressions (3.4). *The following convergences then hold*

$$\upsilon_1^h(\cdot) \to z(\cdot) \text{ in } L_2([0,T]; R^{n_1}) \tag{3.6}$$

$$\upsilon_2^h(\cdot) \to u_*(\cdot) \text{ in } L_2([0,T]; R^{n-n_1}) \text{ when } h \to 0. \tag{3.7}$$

Proof. By virtue of Remarks 1° and 2°, the following estimates hold

$$\sup_{t \in [0,T]} |y(t) - w_1^h(t)|_{n_1} \le \nu^2(h), \quad |\upsilon_1^h(\cdot) - z(\cdot)|_{L_2([0,T]; R^{n_1})} \le \nu(h) = Kh^{1/9}, \tag{3.8}$$

and relation (3.6) follows from these. Hence, in order to prove the theorem, it is necessary to establish the convergence (3.7). In turn, in order to do this, it is sufficient to obtain estimates which are analogous to the estimates from Lemma 1. We shall verify that, when relation (3.5) is satisfied, the following inequalities hold

$$\sup_{t\in[0,T]}\left|z(t)-w_2^h(t)\right|_{n-n_1}^2 \le d_0h^{1/18}, \quad I_{2m}^{(2)}(T)\le J_{2m}(T)+d_1h^{1/36}$$
$$I_{2m}^{(2)}(T) = \int_0^T\left|\upsilon_2^h(\tau)\right|_m^2 d\tau. \tag{3.9}$$

By virtue of expressions (3.4), the relation

$$\left|\upsilon_2^h(t)\right|_m^2 \le 2d^2\alpha^{-2}(\varphi_h^2(t)+\nu_2^h(t)), \tag{3.10}$$

holds where

$$\varphi_h(t) = \left|\upsilon_1^h(t)-z(t)\right|_{n_1}, \quad \mu^h(t) = z(t)-w_2^h(t), \quad \nu_h(t) = \left|\mu^h(t)\right|_{n-n_1}$$

and $d=|D'|$ is the Euclidean norm of the matrix D'. In this case

$$I_{2m}^{(2)}(t)\le 2d^2a^{-2}\int_0^t\{\nu_h^2(\tau)+\varphi_h^2(\tau)\}d\tau. \tag{3.11}$$

It is also obvious that the following inequality holds

$$(D\psi(t),\mu^h(t))_{n-n_1}\le(D\psi(t),\Delta_2^h(t))_{n-n_1}+d\varphi_h(t)\psi_1(t) \quad \text{when} \quad t\in[0,T],$$

where

$$\psi(t) = u_*(t)-\upsilon_2^h(t), \quad \psi_1(t) = |u_*(t)|_m+\left|\upsilon_2^h(t)\right|_m.$$

Further, we have

$$\frac{1}{2}\frac{d\nu_h^2(t)}{dt}\le(D\psi(t),\mu^h(t))_{n-n_1}+L\nu_h^2(t)+L\nu_h(t)\left|y(t)-w_1^h(t)\right|_{n_1}-2L(\Delta_2^h(t),\mu^h(t))_{n-n_1}\le$$
$$\le(D\psi(t),\Delta_2^h(t))_{n-n_1}+d\varphi_h(t)\psi_1(t)+\frac{3}{2}L\nu_h^2(t)+$$
$$+\frac{1}{2}L\left|y(t)-w_1^h(t)\right|_{n_1}^2-2L\nu_2^h(t)+2L\varphi_h(t)\nu_2^h(t).$$

Consequently,

$$\frac{d\nu_h^2(t)}{dt}+\alpha\{\left|\upsilon_2^h(t)\right|_m^2-|u_*(t)|_m^2\}\le\alpha\left|\upsilon_2^h(t)\right|_m^2-2(\upsilon_2^h(t),D'\Delta_2^h(t))_m-\alpha|u_*(t)|_m^2+$$
$$+2(u_*(t),D'\Delta_2^h(t))_m+2d\varphi_h(t)\psi_1(t)+L\nu^4(h)+4L\varphi_h^2(t). \tag{3.12}$$

Without any loss in generality, we will henceforth assume that $\nu(h)\in(0,1)$. Then, on taking account of the rule for determining the control $\upsilon_2^h(\cdot)$ (see relations (3.4)), from inequality (3.12) we obtain

$$\varepsilon^h(t)\le\varepsilon^h(0)+2d\int_0^t\varphi_h(\tau)\psi_1(\tau)d\tau+L(t+4)\nu^2(h), \tag{3.13}$$

where

$$\varepsilon^h(t) = \nu_h^2(t)+\alpha\{I_{2m}^{(2)}(t)-J_{2m}(t)\}.$$

Note that, by virtue of estimates (3.8), the following inequalities hold

$$\int_0^T \varphi_h(\tau)|u_*(\tau)|_m d\tau \le c_0 \nu(\tau)$$

$$2\int_0^T \varphi_h(\tau)\left|\upsilon_2^h(\tau)\right|_m d\tau \le 2\nu(h)\sqrt{I_{2m}^{(2)}(T)} \le \nu^{\beta}(h) + \nu^{2-\beta}(h) I_{2m}^{(2)}(T).$$

In this case, from here and from inequality (3.13), we obtain

$$\varepsilon^h(t) \le \varepsilon^h(0) + c_1(\nu(h) + \nu^2(h) + \nu^{\beta}(h)) + \nu^{2-\beta}(h) I_{2m}^{(2)}(t) \tag{3.14}$$

In turn, by virtue of estimate (3.11), from inequality (3.14) we derive the inequality

$$\varepsilon^h(t) \le \varepsilon^h(0) + c_2 \nu^{2-\beta}(h)\alpha^{-2}\int_0^t \nu_h^2(\tau)d\tau + c_3 G(\alpha), \tag{3.15}$$

where

$$G(\alpha) = \nu^{\beta}(h) + \nu^{4-\beta}(h)\alpha^{-2},$$

and it follows from this that

$$\begin{aligned} \nu_h^2(t) &\le c_4(\varepsilon^h(0) + G(\alpha) + \alpha) + c_2\nu^{2-\beta}(h)\alpha^{-2}\int_0^t \nu_h^2(\tau)d\tau \le \\ &\le c_4(h^2 + G(\alpha) + \alpha)\exp\{c_2 t \nu^{2-\beta}(h)\alpha^{-2}\} \end{aligned} \tag{3.16}$$

(Gronwall's lemma and the equality $\varepsilon^h(0) = h^2$ have been used).

Assuming that, for a certain $\beta \in (0, 1)$ and all $h \in (0, 1)$,

$$\nu^{2-\beta}(h)\alpha^{-2} \le \text{const} \tag{3.17}$$

we obtain from inequality (3.16)

$$\nu_h^2(t) \le c_5(\upsilon^{\beta})(h) + h^{\alpha} + \alpha) \le c_6(h^{\beta/9} + \alpha) \tag{3.18}$$

From relations (3.15) and (3.18), we derive

$$\begin{aligned} \varepsilon^h(t) &\le h^2 + c_7 h^{(2-\beta)/9}\alpha^{-2}(h^{\beta/9} + \alpha) + c_8(h^{\beta/9} + h^{(4-\beta)/9}\alpha^{-2}) \le \\ &\le c_9(H(\alpha) + h^{(4-\beta)/9}\alpha^{-2}) \le c_{10}H(\alpha), \end{aligned}$$

where

$$H(\alpha) = h^{\beta/9} + h^{(2-\beta)/9}\alpha^{-1} + h^{2/9}\alpha^{-2}.$$

Hence,

$$\begin{aligned} & I_{2m}^{(2)}(T) \le J_{2m}(T) + c_{10}\alpha^{-1}H(\alpha) \\ & \left|z(t) - w_2^h(t)\right|_{n-n_1}^2 \le c_{11}(H(\alpha) + \alpha). \end{aligned} \tag{3.19}$$

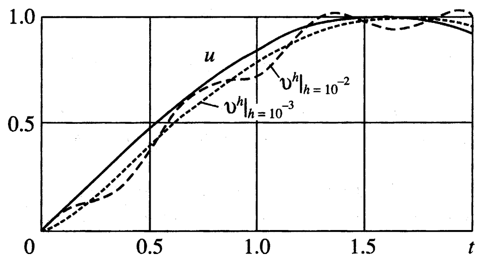

Fig. 1.

Putting $\beta = 3/4$, $\alpha = h^{1/18}$, we obtain estimates (3.9) and (3.10) from estimates (3.19). Note that inequality (3.17) holds for such a choice of α and β.

4. Example

A body moves over an area with known relief under the action of a tractive force $u = u(t)$, $t \in [0, T]$. The gravity force is ignored. Along the path of the motion, approximate data on the phase position of the body are processed. It is required to calculate the tractive force u synchronously with the motion of the body.

We shall consider the simplest model of this situation when a point mass moves along a smooth curve under the action of a force which is directed along the tangent to this curve. The equation of motion has a form similar to that found earlier in Ref. 13:

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = -\beta\eta'(x_1(t)) + u(t). \tag{4.1}$$

Here x_1 is a curvilinear coordinate which determines the position of the point, x_2 is the rate of change of the coordinate, $\eta = \eta(x_1)$ is a smooth function (we assume that it is a Lipschitz function with a Lipschitz constant L), $\eta'(x_1)$ is the derivative of the function $\eta(x_1)$ with respect to x_1 and β is a constant coefficient. We will assume that the time of the motion T is given. The state $x_2(t)$ is measured (with an error) at an instant of time $t \in [0, T]$. The results of the measurements $\xi^h(t)$ have an error h:

$$\left|x_2(t) - \xi^h(t)\right| \le h$$

($|a|$ is the modulus of the number a).

It is required to write out the equation for the continuous estimation of the force $u(t)$.

According to the rule described above, this equation has the form

$$\dot{w}_1^h(t) = w_2^h(t), \quad \dot{w}_2^h(t) = -\beta\eta'(w_1^h(t)) + \upsilon^h(t) + \nu^h(t), \tag{4.2}$$

where

$$\upsilon^h(t) = \alpha^{-1}(\xi^h(t) - w_2^h(t)), \quad \nu^h(t) = \beta L(\xi^h(t) - w_2^h(t)).$$

Systems (4.1) and (4.2) were solved by Euler's method with a step size of 10^{-4} for the case when

$$\alpha = 0.1, \quad x_1(0) = x_2(0) = 0, \quad T = 2, \quad \eta(x_2) = 0$$

The results are shown in Fig. 1 for $h = 10^{-3}$ and $h = 10^{-2}$.

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